Functional Analysis, PhD in Statistics, A.A. 2020-2021

teacher Annalisa Cesaroni

PROGRAM.

Metric spaces. Definition of metric spaces. Topology induced by the metric: balls, open sets, closed sets. Notion of convergence. Complete metric spaces. Continuous functions on metric spaces. Banach Caccioppoli theorem (map contraction theorem). Norms on vectorial spaces, Banach spaces.

Measure theory and integration theory. Definition of σ-algebras, definition of measures, measure spaces. Completion of a σ-algebra. Outer measures and Caratheodory criterion.
Borel σ- algebras and Borel measures. Lebesgue-Stieltjes measures on $\mathbb{R}$, the Lebesgue measure on $\mathbb{R}$ and the Lebesgue measure on $\mathbb{R}^n$. Lebesgue measurable functions. Repartition function associated to a Lebesgue measurable function. Definition of the Lebesgue integral.
Absolutely continuous measure with respect to Lebesgue measure. Example: measure associated to a positive L^1_{loc} function. Example: the Gaussian measure. Integration with respect to a measure which is absolutely continuous w.r.t the Lebesgue measure.
Singular measures with respect to the Lebesgue measure. Example: discrete measures. Integration with respect to discrete measures. The Lebesgue-Radon-Nikodym decomposition.

L^p spaces. Spaces of integrable functions, L^p spaces. The Chebychev inequality, the Young inequality and the Hölder inequality, with applications. Monotone convergence theorem, Fatou lemma and dominated convergence theorem.

Modes of convergence and approximations in L^p. Different modes of convergence for sequences of functions: convergence in measure, almost everywhere convergence, convergence in L^p, weak convergence in L^p. Relations between the different modes of convergence. Weak convergence of measures (some hint).
Simple functions are dense in L^p. Convolution in L^p, Young inequality. Mollifiers and approximated identity. C^∞ functions are dense in L^p.
Rescaled periodic functions converges weakly to their average, Riemann-Lebesgue lemma.
Central limit theorem.

Textbook.

- Lecture notes by the teacher (and references therein).
- G. B. Folland *Real Analysis: modern tecniques and their applications.* Wiley 1999 (2nd ed)