Multidimensional Inequality Measures on Finite Partial Orders

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Goal

To outline a new and alternative conceptual and mathematical approach to the measurement of inequality in multidimensional systems of ordinal variables.

Alternative to what? To classical attempts that address ordinal variables with the same conceptual tools used for cardinal variables.

Why new? Because, as far as I know, it is the first time that the issue is addressed based on the theory of partial orders and Hasse diagrams.

The presentation will be mainly conceptual, since it would take too long to work out all the mathematical details.
The problem
Multidimensional ordinal inequality: why?

Poverty, deprivation, well-being quality-of-life...

- Monetary
  - Unidimensional inequality
- Non-monetary («beyond GDP»)
  - Multidimensional inequality (often with ordinal variables)
For example

Consider surveys like EU-SILC and ask:

Is multidimensional deprivation increasing in Italy?

Considering:
• Ownership of goods (yes/no).
• Access to services (different difficulty levels).
• Personal economic/labour situation (from easy to hard).
• , , ,
## Inequality measures: state of the art

<table>
<thead>
<tr>
<th></th>
<th>Unidimensional</th>
<th>Multidimensional</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinal</td>
<td>OK</td>
<td>OK</td>
</tr>
<tr>
<td>Ordinal</td>
<td>OK</td>
<td>?</td>
</tr>
</tbody>
</table>
1. Define a criterion to state when a unidimensional frequency distribution on a given population is more unequal than another: Examples: Pigou–Dalton criterion (Lorenz curve), for cardinal variables; Allison–Foster criterion for ordinal variables.

2. Derive unidimensional index of inequality, based on axioms comprising consistency to the selected criterion of inequality comparison.

Results: a lot of inequality indices for both the cardinal and the ordinal case.
Unidimensional indices of ordinal inequality

<table>
<thead>
<tr>
<th>Name</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abul Naga – Yalcin</td>
<td>Kobus – Milos</td>
</tr>
<tr>
<td>Allison – Foster</td>
<td>Kvalseth</td>
</tr>
<tr>
<td>Apouey</td>
<td>Leti</td>
</tr>
<tr>
<td>Berry – Mielke</td>
<td>Reardon 1</td>
</tr>
<tr>
<td>Blair – Lacy 1</td>
<td>Reardon 2</td>
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<tr>
<td>Blair – Lacy 2</td>
<td>Reardon 3</td>
</tr>
<tr>
<td>Cowell – Flachaire</td>
<td>Reardon 4</td>
</tr>
<tr>
<td>Leik</td>
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</tr>
</tbody>
</table>

NB. Some of these indices are in fact the same formula rediscovered many times, by different authors.
Example: Normalized Leti

Let $V$ be an ordinal variable with $M$ degrees and let $p_1, \ldots, p_M$ be the percentage (relative frequency) of population sharing degree 1, \ldots, degree $M$.

$$I = \frac{2}{M - 1} \sum_{i, j = 1}^{M} p_i p_j |i - j|$$
Multidimensional case: classical «logic»

1. Consider k variables and the correspondent joint frequency distribution.

2. Extend the Pigou–Dalton or the Allison–Foster criterion to the multidimensional case

3. Derive multidimensional indices of inequality, based on axioms comprising consistency to the selected criterion of inequality comparison.

Results: in the cardinal case, families of indices. In the ordinal case, some preliminary (yet non-satisfactory) proposals.
The difficult point in extending the unidimensional approach to the multidimensional case is that \textit{it is not clear how to compare two multidimensional distributions in terms of inequality}, i.e. it is not clear how to extend the Pigou–Dalton or the Allison–Foster criterion.

In the ordinal case, there is a supplementary difficulty: how to manage non-numerical attributes mathematically?
The deep conceptual problem

Implicitly or not, the basic idea underlying multidimensional ordinal indices is to aggregate unidimensional inequality measures, relative to the single attributes. Here the problem is how the joint distribution is taken into account, to avoid getting just an average of unidimensional inequality measures.

The results are indeed very cumbersome.

So two alternatives are left:
1. To develop the classical approach in some more sophisticated way.
2. To change perspective and address the problem in another way, through more suitable mathematical tools:

PARTIALLY ORDERED SETS
Partially ordered sets
Let $X$ be a (here, finite) set and let $R \subseteq X \times X$ be a binary relation on $X$. If $R$ satisfies:

1. $(x, x) \in R$, $\forall x, \in X$ (reflexivity);
2. $(x, y) \in R$ and $(y, x) \in R \iff x = y$ (antisymmetry);
3. $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ (transitivity);

then $R$ is a partial order, in symbols: $\leq$.

$(X, \leq)$ is a partially ordered set or, for short, a poset.
1) \( x \leq x, \ \forall \ x \in X \) (reflexivity);

2) \( x \leq y \) e \( y \leq x \) \( \Leftrightarrow \) \( x = y \) (antisymmetry);

3) \( x \leq y \) e \( y \leq z \) \( \Rightarrow \) \( x \leq z \) (transitivity).
Hasse diagram
Hasse diagram – comparability

Comparable elements
Hasse diagram – incomparability

Incomparable elements
Hasse diagram – comparability

Covering
Hasse diagram – maximum
Hasse diagram – minimal elements
Hasse diagram – chain
Hasse diagram – antichain
Hasse diagram – downset
Hasse diagram – upset
Lattice

\[ a \lor (\text{«join»}) \]

\[ a \land b \ (\text{«meet»}) \]
Join – semilattice

$\lor$
Meet – semilattice

\[ a \land b \ (\text{«meet»}) \]
Linear order (aka: complete or total order)
Extension of a poset
Extensions of a poset
Linear extensions

\[ \subseteq \]
An important property

\[ a \cap b \cap c \cap d \cap e = \]

\[ \begin{align*}
  &\quad a \\
  &b \\
  &\quad c \\
  &d \\
  &e \\
  &\quad a \\
  &b \\
  &c \\
  &d \\
  &e \\
  &\quad a \\
  &b \\
  &c \\
  &d \\
  &e \\
\end{align*} \]
Poset dimension

\[ \text{Poset dimension} = \bigcap \]

Diagram of a poset with vertices labeled a, b, c, d, e.
Example: let $V_1$, $V_2$, $V_3$ be three 0/1 variables pertaining to the ownership of three different goods (e.g. home, car, washing machine). The set of 8 «ownership profiles» is naturally a partially ordered set, with the following Hasse diagram:

1 stands for «ownership»

0 stands for «non-ownership»
The general theory
Let $V_1, \ldots, V_k$ be $k$ ordinal variables (possibly with a different number of degrees).

Let $S$ be the poset of profiles (combinations of variable scores) partially ordered according to product order.

Let $P$ be a (relative) frequency distribution on the profiles of $S$.

How can we assess the degree of inequality of $P$ on $S$?
How much unequal is this distribution?

- statistical unit
As common in mathematics, sometimes it is easier to get the solution to this problem, considering it at a more abstract level.

So we start from the following issue:

Given a frequency distribution on a finite poset (not necessarily coming from unidimensional variables as a product order), how can we measure its inequality?
By example
When posets are considered, there are two key aspects that must be taken into account to get consistent inequality measures:

1. How the measure changes as the distribution changes (consistency to Allison–Foster principle or its possible extensions).
2. How the measure changes when the poset is extended. That is, when its structure changes.

Up to now, focus has been on point 1 (with the problems already discussed). We instead focus on point 2.
Recall that poset S is part of a larger structure…
Given the distribution over the elements of the set underlying the posets of the semilattice:

*can we assign inequality measures to the different posets in an independent manner? or there must be some consistency rules linking the value of an inequality index on poset $S$ to its values on extensions of $S$?*
Outline of the poset – based axiomatic approach

1. Poset $S$ is linked to its extensions through intersections: if one knows extensions of $S$, one can reconstruct it considering common comparabilities.
2. Thus, it is natural to state that an inequality measure on $S$ must be a function of inequality measures on extensions of $S$.
3. Axioms are identified so as to assure that this functional relationship is meaningful.
4. It turns out that these axioms lead to the classical Kolmogorov–Nagumo–De Finetti theorem: the inequality measure on $S$ is some kind of mean of inequalities on (special classes of) its extensions.
5. In particular, it is a mean of inequality measures on linear extensions.
6. But linear extensions are simple linear orders and we can measure inequality on them using any unidimensional index.
7. So we can reconstruct inequality on a poset, by unidimensional inequalities on its linear extensions.
1. Our axiomatic approach is a theory of how to aggregate unidimensional indices to get inequality measures on posets.

2. We do consider explicitly the problem of extending the Allison–Foste criterion: unidimensional indices are consistent with it so «they assure for it». 
To run the computations, we must preliminarly select the unidimensional index chosen to assess inequality on linear extensions and the aggregation formula (which mean?):

Unidimensional index: normalized Leti

\[ I = \frac{2}{M - 1} \sum_{i,j=1}^{M} p_i p_j |i - j| \]

Aggregation: average
Example

«Grey» posets stand for the value of the inequality measure on the «white» posets, given the frequency distribution.
Example

\[
\frac{1}{5} \times (0.728 + 0.79 + 0.79 + 0.78 + 0.64 + 0.64) = 0.728
\]
Different representation and consistency

\[=\]
Example

$$= \frac{1}{5} \times \left( \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} + \begin{array}{c} a \\ b \\ d \\ c \\ e \end{array} + \begin{array}{c} a \\ c \\ b \\ d \\ e \end{array} + \begin{array}{c} b \\ a \\ c \\ d \\ e \end{array} + \begin{array}{c} b \\ a \\ d \\ c \\ e \end{array} \right)$$
Example

\[
\begin{align*}
\text{Diagram 1} & \quad = \quad \frac{3}{5} \times \text{Diagram 2} + \frac{2}{5} \times \text{Diagram 3} \\
0.728 & \quad \text{Diagram 1} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Another example
Example

\[ \frac{1}{5} \times \]
Example

\[ \begin{align*}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array}
\end{align*} = \frac{4}{5} \times \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array} + \frac{1}{5} \times \begin{array}{c}
\text{Diagram 3}
\end{array} \]
Example

\[
\begin{align*}
\text{Diagram 1} & \quad = \quad \frac{4}{5} \times \quad \text{Diagram 2} \quad + \quad \frac{1}{5} \times \quad \text{Diagram 3} \\
\text{Diagram 1} & \quad = \quad 0.728 \\
\text{Diagram 2} & \quad = \quad \frac{4}{5} \times \quad 0.715 \\
\text{Diagram 3} & \quad = \quad \frac{1}{5} \times \quad 0.78 \\
\text{Combined} & \quad = \quad 0.728
\end{align*}
\]
Final results

0.79

0.79

0.78

0.64

0.64

0.79

0.79

0.785

0.785

0.71

0.715

0.786

0.728
Multidimensional inequality indices
The theory of multidimensional inequality indices on k ordinal variables is simply the theory just outlined, applied to product orders.

So the procedure is:

1. Given k ordinal variables, built the product order of the corresponding profiles.
2. Choose an unidimensional index.
3. Choose an aggregation method.
4. Compute…
Example from the Multipurpose Survey on Families (Istat)

Year 2010  
Country: Italy  
Data: 3 dichotomized variables pertaining to subjective satisfaction on one’s own 1) economic situation, 2) health and 3) leisure time.

<table>
<thead>
<tr>
<th>Profile</th>
<th>Counts</th>
</tr>
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<tbody>
<tr>
<td>111</td>
<td>14207</td>
</tr>
<tr>
<td>110</td>
<td>4255</td>
</tr>
<tr>
<td>101</td>
<td>1291</td>
</tr>
<tr>
<td>011</td>
<td>9282</td>
</tr>
<tr>
<td>100</td>
<td>807</td>
</tr>
<tr>
<td>010</td>
<td>5876</td>
</tr>
<tr>
<td>001</td>
<td>2400</td>
</tr>
<tr>
<td>000</td>
<td>2847</td>
</tr>
</tbody>
</table>

1: good  
0: bad
Multidimensional inequality 0.72

Example

Distribution of inequality measures over the set of 48 linear extensions of the poset.
Towards attribute decomposition

An important issue in multidimensional inequality measurement is to decompose the overall measure into a sum of terms, some of which may be interpreted as the contribution due to unidimensional inequalities on each single variable.

Poset based approach offers an elegant solution to this issue, that we outline briefly in the following.

The starting point is to recall that we can represent a poset as intersection of its extensions in different ways. We draw upon this degree of freedom to choose the representation more useful to address attribute decomposition.

We outline the main ideas through an example.
An useful representation
An useful representation

«Conditional» posets

«Compensative» poset
An useful representation

«Conditional» posets

«Compensative» poset
An useful representation

REMARK. This is not an attribute decomposition, in fact it is valid in general, irrespective of the chosen unidimensional index. Nevertheless, it reminds of an attribute decomposition...
But if the unidimensional index has «good» properties (and the Leti index has), it turns out that:

\[ \begin{align*}
111 &\quad 0.. \\
110 &\quad 1.. \\
100 &\quad + \text{ other terms} \\
011 & \\
010 & \\
001 & \\
000 &
\end{align*} \]

And similarly for the other conditional posets.
Moreover, it also turns out that

Inequality relative to the number of «unsatisfaction».

\[ \text{Sum} = 1 \quad \text{Sum} = 0 \quad + \quad \text{other terms} \]
Putting things together

Contribution of attribute inequalities

\[ = \frac{1}{12} \times 1.. + \frac{1}{12} \times .1. + \frac{1}{12} \times ..1 + \frac{9}{12} \times \text{Sum} = 0 + \text{Sum} = 1 + \text{Sum} = 2 + \text{Sum} = 3 \]

+ other terms (interactions)
Computational aspects
To perform all the computations in practice, one should list all of the linear extensions of the poset at hand.

With real posets this is impossible, so one must sample from the set of linear extensions. There are efficient algorithms to do this, particularly the Bubly–Dyer algorithm.

At the same time, one should also estimate the coefficients of the weighted sum.

So the research field is completely open at theoretical, computational and practical levels.
Thanks
Some author’s publications


• Fattore M. “Hasse diagrams, poset theory and fuzzy poverty measures”, Rivista Internazionale di Scienze Sociali 1/2008, Vita e Pensiero, Milano, 2008 ISSN 0035-676X.